JOURNAL OF
PURE AND
APPLIED ALGEBRA

ELSEVIER Journal of Pure and Applied Algebra 128 (1998) 171–183

Coflat monomorphisms of coalgebras

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Communicated by C.A. Weibel; received 18 October 1995; received in revised form 28 October 1996

Abstract

We consider left coflat monomorphisms of coalgebras, and establish a 1–1 correspondence between the set of isomorphism classes of left coflat monomorphisms, the set of some coidempotent subcoalgebras and the set of equivalence classes of perfect localization bicomodules as well. © 1998 Elsevier Science B.V. All rights reserved.

1991 Math. Subj. Class.: 16W24

0. Introduction

In this paper, we consider coflat monomorphisms of coalgebras. These are dual to the perfect localizations of algebras (or the flat epimorphisms of algebras). If $\phi : C \longrightarrow D$ is a left coflat monomorphism of coalgebras, then ϕ determines a Morita–Takeuchi context

$$(C, D, {}_CU_D, {}_DC_C, f, g)$$

and the bicolinear map f is an isomorphism. It follows that C is quasi-finite as a left D-comodule, and the coendomorphism coalgebra of the left D-comodule ${}_DC$ is canonically isomorphic to C (cf. Theorem 3.4 and Corollary 3.5). It has been shown in [2] that a left coflat monomorphism $\phi : C \longrightarrow D$ of coalgebras determines a hereditary torsion theory $\mathrm{Ker}(-)^{\phi}$, $(-)^{\phi} = - \square_D C$, of the comodule category $\mathbf{M}^D$, and any hereditary torsion theory is uniquely determined by a coidempotent subcoalgebra A of D. It is natural to ask what conditions on A allow to reconstruct the coflat monomorphism ϕ. This leads us to define localization bicomodules of a coalgebra D in Section 2. We

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[1] Supported by a grant from NATO and PB91-706 from DGICYT.

PII S0022-4049(97)00042-X

show that coidempotent subcoalgebras of D bijectively correspond to localization bicomodules. This correspondence yields a 1–1 correspondence between left coflat monomorphisms and the so-called left perfect localizations which precisely answers the question.

1. Preliminaries

Throughout, k is a fixed field. All coalgebras, algebras, vector spaces, and unadorned $\otimes$, Hom, etc. are over k. Throughout, Λ, Γ, C and D always stand for coalgebras. The character $\mathbf{M}$ indicates the category of k-modules. We refer to [4] for detail of coalgebras and comodules. If C is a coalgebra, we denote by $\mathbf{M}^C$ the category of right C-comodules. Similarly, we let ${}^C\mathbf{M}$ stand for the left C-comodule category. A right C-comodule X is injective (or C-injective) if the functor $\mathrm{Com}_{-C}(-,X)$ is exact.

A C–D-bicomodule is a left C-comodule and a right D-comodule X, denoted by ${}_CX_D$, such that the C-comodule structure map $\rho_C : X \longrightarrow C\otimes X$ is D-colinear, or equivalently the D-comodule structure map $\rho_D : X \longrightarrow X \otimes D$ is C-colinear. In particular, C is a C–C-bicomodule through Δ.

Cotensor product. For a right C-comodule M and a left C-comodule N, the cotensor product $M \,\square_C\, N$ is the kernel of

$$\rho_M \otimes 1 - 1 \otimes \rho_N : M \otimes N \rightrightarrows M \otimes C \otimes N.$$

The functors $M \,\square_C -$ and $- \,\square_C N$ are left exact and preserve direct sums. If ${}_\Lambda X_C$ and ${}_C Y_\Gamma$ are bicomodules, then $X \,\square_C\, Y$ is a $\Lambda - \Gamma$-bicomodule induced by the structure maps: $\rho_\Lambda : X \longrightarrow \Lambda\otimes X$ and $\rho_\Gamma : Y \longrightarrow Y\otimes\Gamma$. The cotensor product is associative. For comodules X_C and ${}_CY$ the structure maps ρ_X and ρ_Y induce C-colinear isomorphisms $X \simeq X \,\square_C\, C$ and $Y \simeq C \,\square_C Y$. If X is a right C-comodule which is finite dimensional as vector space, then the dual X^* is a left C-comodule with structure map

$$X^* \longrightarrow \mathrm{Com}_{-C}(X,C) \hookrightarrow \mathrm{Hom}(X,C) \simeq C \otimes X^*, \quad x^* \mapsto (x^* \otimes 1)\rho_X.$$

If Y is a right C-comodule, then we have the canonical isomorphism

$$Y \square_C X^* \simeq \mathrm{Com}_{-C}(X,Y). \tag{1}$$

If, moreover, Y is a D–C-bicomodule, then $\mathrm{Com}_{-C}(X,Y)$ is a left D-comodule induced by (1). A right C-comodule Y is called a coflat comodule if the functor $Y \square_C -$ is exact. Since every comodule is the union of its finite-dimensional subcomodules, it follows from (1) that Y_C is coflat if, and only if, $\mathrm{Com}_{-C}(-,Y)$ is exact if and only if Y is C-injective, cf. [6].

Co-hom functor. A comodule X_C is quasi-finite if $\mathrm{Com}_{-C}(Y,X)$ is finite-dimensional for every finite-dimensional comodule Y_C. We recall from [5] the definition of the co-hom functor and some of its basic properties.

Basic lemma. *Let $_CX_D$ be a bicomodule. Then X_D is quasi-finite if and only if the functor $-\square_C X : \mathbf{M}^C \longrightarrow \mathbf{M}^D$ has a left adjoint functor, denoted by $h_{-D}(X,-)$. That is, for comodules Y_D and W_C,*

$$\mathrm{Com}_{-C}(h_{-D}(X,Y),W) \simeq \mathrm{Com}_{-D}(Y, W \square_C X). \tag{2}$$

Assume that X_D is a quasi-finite comodule, then $e_{-D}(X) = h_{-D}(X,X)$ is a coalgebra, called the co-endomorphism coalgebra of X. The comultiplication of $e_{-D}(X)$ corresponds to $(1 \otimes \theta)\theta : X \longrightarrow e_{-D}(X) \otimes e_{-D}(X) \otimes X$ in (2) when $C = k$, and the counit of $e_{-D}(X)$ corresponds to the identity map 1_X. X is an $e_{-D}(X)-D$-bicomodule with the left comodule structure map θ given by the canonical map $X \longrightarrow h_{-D}(X,X) \otimes X$.

Morita–Takeuchi (M–T) context. An M–T context $(C, D, {}_CP_D, {}_DQ_C, f, g)$ consists of coalgebras C, D, bicomodules $_CP_D$, $_DQ_C$, and bicolinear maps $f: C \longrightarrow P \square_D Q$ and $g: D \longrightarrow Q \square_C P$ satisfying the following commutative diagrams:

$$\begin{array}{ccc} P & \xrightarrow{\ \sim\ } & P\square_D D \\ \big\downarrow{\scriptstyle\sim} & & \big\downarrow{\scriptstyle 1\square g} \\ C\square_C P & \xrightarrow{f\square 1} & P\square_D Q\square_C P \end{array} \qquad \begin{array}{ccc} Q & \xrightarrow{\ \sim\ } & Q\square_C C \\ \big\downarrow{\scriptstyle\sim} & & \big\downarrow{\scriptstyle 1\square f} \\ D\square_D Q & \xrightarrow{g\square 1} & Q\square_C P\square_D Q \end{array}$$

The context is said to be *strict* if both f and g are injective (equivalently, isomorphic). In this case, we say that C is M–T equivalent to D. Let P_D be a quasi-finite comodule and $C = e_{-D}(P)$. Then $_CP_D$ is a bicomodule. Set $_DQ_C = h_{-D}(P,D)$, $g = \theta : D \longrightarrow Q \square_C P$, and $f : C \cong h_{-D}(P, P \square_D D) \xrightarrow{\partial} P \square_D h_{-D}(P,D) = P \square_D Q$. Then $(C, D, {}_CP_D, {}_DQ_C, f, g)$ is an M–T context, where f is injective if and only if P_D is injective, and g is injective if and only if P_D is a cogenerator in $\mathbf{M}^D$, cf. [5].

Let $\phi : C \longrightarrow D$ be a coalgebra map. Every right C-comodule X may be viewed as a right D-comodule with the structure map

$$(1 \otimes \phi)\rho : X \longrightarrow X \otimes C \longrightarrow X \otimes D.$$

In this case, we will say that X_C restricts to the right D-comodule X_D. The map ϕ induces a left exact (restriction) functor:

$$(-)_\phi : \mathbf{M}^C \longrightarrow \mathbf{M}^D.$$

Let us recall from [2] the relation between monomorphisms of coalgebras and torsion theories in a comodule category. A coalgebra map $\phi : C \longrightarrow D$ is said to be a monomorphism if it is a monomorphism in the coalgebra category $\mathbf{Cog}_k$. Let $(-)^\phi$ be the cotensor functor

$$\mathbf{M}^D \longrightarrow \mathbf{M}^C, \quad M \mapsto M \,\square_D\, C.$$

Theorem 1.1 (Nastasescu and Torrecillas [2, Theorem 3.5]). *Let $\phi\ C \longrightarrow D$ be a coalgebra map. The following are equivalent*:

(1) *ϕ is a monomorphism in $\mathbf{Cog}_k$.*
(2) *$C \,\square_D\, \mathrm{Ker}\phi = 0$.*
(3) *The canonical morphism $\overline{\Delta} : C \longrightarrow C \,\square_D\, C$ is an isomorphism.*
(4) *The restriction functor $(-)_\phi : \mathbf{M}^C \longrightarrow \mathbf{M}^D$ is full.*
(5) *The canonical functorial morphism $I_{\mathbf{M}^C} \longrightarrow (-)^\phi \circ (-)_\phi$ is an isomorphism.*

Note that conditions (4) and (5) in the above theorem may be replaced by the left comodule versions since condition (3) is symmetric. Let D be a coalgebra, $\mathbf{M}^D$ the comodule category. A subcategory $\mathscr{C}$ of $\mathbf{M}^D$ is a closed subcategory if $\mathscr{C}$ is closed under subobjects, quotient objects and direct sums. If, in addition, $\mathscr{C}$ is closed under extensions, then $\mathscr{C}$ is called a localizing subcategory. We refer to [3] for detail on (hereditary) torsion theories. A subcoalgebra A of D is said to be coidempotent if $A = A \wedge A = \mathrm{Ker}(D \xrightarrow{\Delta} D/A \otimes D/A)$.

Theorem 1.2 (Nastasescu and Torrecillas [2, Theorems 4.2, 4.5]). *Let D be a coalgebra and A be a subcoalgebra of D. We denote by $\mathscr{T}_A = \{M \in \mathbf{M}^D | \rho_M(M) \subseteq M \otimes A\}$. Then*

(1) *$\mathscr{T}_A$ is a closed subcategory of $\mathbf{M}^D$.*

(2) *The map $A \mapsto \mathscr{T}_A$ is a bijective map between the set of all subcoalgebras of D and the set of all closed subcategories of $\mathbf{M}^D$.*

(3) *$A \mapsto \mathscr{T}_A$ gives an one-to-one correspondence between the set of coidempotent subcoalgebras of D and the set of localizing subcategories of $\mathbf{M}^D$.*

(4) *All the localizing subcategories of $\mathbf{M}^D$ are hereditary torsion theories.*

Note that the theorem still holds if one considers the left comodule category ${}^D\mathbf{M}$.

Let $\phi : C \longrightarrow D$ be a coalgebra map. ϕ is said to be a left coflat monomorphism if ϕ is a monomorphism and the comodule ${}_DC$ is coflat. Let ϕ be a left coflat monomorphism. The canonical functor:

$$(-)^\phi : \mathbf{M}^D \longrightarrow \mathbf{M}^C, \quad X \mapsto X \,\square_D\, C$$

is an exact functor that commutes with direct sums. It follows that the kernel $\mathrm{Ker}(-)^\phi = \mathscr{T}$ is a localizing subcategory of $\mathbf{M}^D$. By [2, Theorem 4.5] there exists a unique coidempotent subcoalgebra A of D such that

$$\mathscr{T} = \{M \in \mathbf{M}^D | \rho(M) \subseteq M \otimes A\} = \mathbf{M}^A.$$

Let us denote it by $\mathscr{T}_A$. Since $\mathscr{T}_A$ is closed under products $\mathscr{T}_A$ is a hereditary torsion theory and it is a TTF class. Note that A is a subcoalgebra of D. Hence, ${}_A\mathscr{T}$ is a hereditary torsion theory and a TTF class in ${}^D\mathbf{M}$.

2. Localization bicomodules

In this section, we define (perfect) localizations and show that any left coflat monomorphisms $\phi : C \longrightarrow D$ of coalgebras comes from some coidempotent subcoalgebra of D. There is ono-to-one correspondence between the set of left coflat monomorphisms to D and the set of equivalence classes of perfect localization bicomodules.

Let D be a coalgebra. By a *localization bicomodule*, we mean a pair (U, ψ) of a D-bicomodule U and a D-bicomodule map $\psi : D \longrightarrow U$ such that $U \square_D \psi$ and $\psi \square_D U$ are isomorphisms.

First, we establish a correspondence between localization bicomodules of a coalgebra D and coidempotent subcoalgebras of D.

Lemma 2.1. *Let (U, ψ) be a localization bicomodule of D. Then* $\operatorname{Ker}\psi$ *is a coidempotent subcoalgebra of D.*

Proof. $A = \operatorname{Ker}\psi$ is a subcoalgebra since ψ is D-bicolinear. Let $\mathscr{T}_A$ be the category $\mathbf{M}^A$ of right A-comodules. If $X \in \mathbf{M}^D$, then $X \in \mathscr{T}_A$ if and only if $X \square_D U = 0$. Indeed, if $X \in \mathscr{T}_A$, then $X \square_D U \simeq X \square_A A \square_D U = 0$ since $A \square_D U = 0$. Conversely, if $X \square_D U = 0$, then $X \square_D A \simeq X$, and $X \in \mathscr{T}_A$. Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is an exact sequence in $\mathbf{M}^D$ such that X, Z are in $\mathscr{T}_A$, then $X \square_D U = 0 = Z \square_D U$. This implies that $Y \square_D U = 0$, and Y is in $\mathscr{T}_A$. This means that $\mathscr{T}_A$ is closed under extension. By [2, Theorem 4.5] A is a coidempotent subcoalgebra of D. $\square$

Lemma 2.2. *Let (U, ψ) be a localization D-bicomodule, A the coidempotent subcoalgebra* $\operatorname{Ker}\psi$. *If $X \xrightarrow{f} Y$ is a left (or right) D-comodule map, then* $\operatorname{Ker} f, \operatorname{Coker} f$ *are in ${}_A\mathscr{T}$ (or $\mathscr{T}_A$) if and only if $U \square_D f$ (or $f \square_D U$) is an isomorphism.*

Proof. By the argument in the proof of Lemma 2.1, a D-comodule is torsion (or in ${}_A\mathscr{T}$) iff $U \square_D X = 0$. Since $\operatorname{Ker} f$ is torsion, we have $U \square_D \operatorname{Ker} f = 0$ and hence, $U \square_D f$ is injective. To show that $U \square_D f$ is surjective, we may assume thet f is surjective since $U \square_D f(X) = U \square_D Y$ (because $U \square_D \operatorname{Coker} f = 0$). Now, the map

$$\psi \square_D X : X \longrightarrow U \square_D X$$

restricts to zero on $\operatorname{Ker} f$ since $\operatorname{Ker} f$ is torsion and $U \square_D X$ is torsion free as left D-comodules. Hence, it factors through f, and we may write

$$\psi \square_D X = h \circ f, \quad h : Y \longrightarrow U \square_D X.$$

Thus, we obtain

$$\psi \square_D Y = (U \square_D f) \circ h : Y \longrightarrow U \square_D Y.$$

Hence, $U \square_D \psi \square_D Y = (U \square_D U \square_D f) \circ (U \square_D h)$. But this is an isomorphism since $U \square_D \psi$ is. It follows that $U \square_D U \square_D f$ is surjective. Since $U \simeq U \square_D U$ as

D-bicomodule, we obtain that $U\square_D f$ should be surjective, and we conclude that $U\square_D f$ is an isomorphism.

Now, let $0 \longrightarrow \mathrm{Ker} f \longrightarrow X \xrightarrow{f} Y \longrightarrow \mathrm{Coker}\, f \longrightarrow 0$ be an exact sequence of left D-comodules such that $U\square_D f$ is an isomorphism. Since $U\square_D -$ is a left exact functor $U\square\,\mathrm{Ker} f = 0$, that is, $\mathrm{Ker} f$ is torsion. For any object $X \in {}^D\mathbf{M}$, we have a left D-colinear map $\psi_X = \psi\square_D X : X \longrightarrow U\square_D X$. Since $U\square_D \psi_X$ is an isomorphism, $\mathrm{Ker}\,\psi_X$ is torsion. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 = & U\square_D \mathrm{Ker} f & \longrightarrow & U\square_D X & \underset{\sim}{\xrightarrow{U\square f}} & U\square_D Y & \xrightarrow{U\square p} & U\square_D \mathrm{Coker}\, f & \\
 & \uparrow{\scriptstyle \psi_{\mathrm{Ker} f}} & & \uparrow{\scriptstyle \psi_X} & & \uparrow{\scriptstyle \psi_Y} & & \uparrow{\scriptstyle \psi_{\mathrm{Coker}\, f}} & \\
0 \longrightarrow & \mathrm{Ker} f & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{p} & \mathrm{Coker}\, f & \longrightarrow 0
\end{array}$$

Since $U\square_D f$ is an isomorphism, $U\square_D p$ should be zero. It follows from the above diagram that $\psi_{\mathrm{Coker}\, f}$ is zero. But the kernel of $\psi_{\mathrm{Coker}\, f}$ is torsion, and hence, $\mathrm{Coker}\, f$ is torsion. $\square$

For hereditary torsion theory $\mathscr{T}_A$, one may form a quotient category $\mathbf{M}^D/\mathscr{T}_A$, denoted by $\overline{\mathbf{M}^D}$. Let (T_A, S_A) be an adjoint pair of canonical functors. We have

Lemma 2.3. *The section functor* $S_A : \mathbf{M}^D/\mathscr{T}_A \longrightarrow \mathbf{M}^D$ *preserves direct sums and quasi-finiteness.*

Proof. Let $\{X_i\}$ be a family of objects in $\mathbf{M}^D/\mathscr{T}_A$. We have the canonical map

$$0 \longrightarrow \bigoplus_i S_A(X_i) \xrightarrow{\alpha} S_A(\bigoplus_i X_i),$$

where α is injective since $\mathrm{Ker}\,\alpha$ is torsion and $\bigoplus_i S_A(X_i)$ is torsion free. Suppose that α is not surjective. Since $\mathbf{M}^D$ is locally Noetherian, there exists a set of Noetherian generators $\{V_j\}$. If the monomorphism α is not epic, then there is some $V_l \in \{V_j\}$ and a non-zero morphism $f : V_l \longrightarrow S_A(\bigoplus_i X_i)$ such that f cannot factor through α. However, since V_l and $T_A(V_l)$ are Noetherian, we have

$$\begin{aligned}
\mathrm{Com}_{-D}(V_l, S_A(\bigoplus_i X_i)) &\simeq \mathrm{Hom}(T_A(V_l), \oplus_i X_i) \\
&\simeq \bigoplus_i \mathrm{Hom}(T_A(V_l), X_i) \\
&\simeq \bigoplus_i \mathrm{Com}_{-D}(V_l, S_A(X_i)) \\
&\simeq \mathrm{Com}_{-D}(V_l, \bigoplus S_A(X_i)),
\end{aligned}$$

where Hom means the Hom in $\mathbf{M}^D/\mathscr{T}_A$. It follows that f factors through α, a contradiction. So α is an isomorphism. Note that S_A is a right adjoint functor of T_A and T_A preserves objects of finite dimensions. These facts yield that S_A respects quasi-finiteness. $\square$

Lemma 2.4. *Let A be a coidempotent subcoalgebra of D. Then $S_AT_A(D)$ together with the canonical adjunction map $\psi: D \longrightarrow S_AT_A(D)$ is a localization bicomodule. Moreover, by symmetry, $({}_AS_AT(D), \psi')$ is a localization bicomodule and there is a bicolinear isomorphism $\theta: S_AT_A(D) \longrightarrow {}_AS_AT(D)$ such that $\psi' = \theta \circ \psi$.*

Proof. Let A be a coidempotent subcoalgebra of D. The localization functor $S_AT_A: \mathbf{M}^D \longrightarrow \mathbf{M}^D$ is a left exact functor and preserves direct sums by Lemma 2.3. So it is of form $- \square_D U$ for some D-bicomodule U by [5, 2.1]. In this case, the adjunction $\psi: I_D \longrightarrow S_AT_A$ is represented by a D-bicomodule map $\psi: D \longrightarrow U$. Since a comodule X is torsion if and only if $S_AT_A(X) = 0$, we obtain that X is torsion iff $X \square_D U = 0$. Now $\operatorname{Ker}\psi$ and $\operatorname{Coker}\psi$ are torsion. By Lemma 2.2, we obtain that $\psi \square_D U$ is an isomorphism. To show $U \square_D \psi$ is also an isomorphism, we consider the difference map

$$f = U \square_D \psi - \psi \square_D U : U \longrightarrow U \square_D U,$$

which is obviously right and left D-colinear. It is clear that $f \circ \psi = 0$. So f factors through $\operatorname{Coker}\psi$ which is torsion. Since $U \square_D U$ is torsion free as a right D-comodule, any right D-colinear map from $\operatorname{Coker}\psi$ to $U \square_D U$ should be zero. It follows that $f = 0$. Therefore (U, ψ) is a localization

By symmetry, $({}_AS_AT(D), \psi)$ is a localization bicomodule. Let U' be ${}_AS_AT$. Since $\operatorname{Ker}\psi'$ and $\operatorname{Coker}\psi'$ are in ${}_A\mathscr{T}$, By Lemma 2.2, we obtain that $U \square_D \psi: U \longrightarrow U \square_D U'$ is a bicolinear isomorphism. By symmetry, $\psi \square U'$ is a bicolinear isomorphism too. Let θ be $(\psi \square U')^{-1} \circ (U \square \psi')$. Then $\theta: U \longrightarrow U'$ is a bicolinear isomorphism such that $\psi' = \theta \circ \psi$. $\square$

From the proof of Lemma 2.4, we obtain that $\psi \square U = U \square \psi$ if (U, ψ) is a localization bicomodule. Two localizations (U, ψ) and (U', ψ') are *equivalent* if there exists an D-bicolinear isomorphism $\mu: U \longrightarrow U'$ such that $\psi = \psi'\mu$. Let Ł be the set of equivalence classes of localization bicomodules of coalgebra D. Denote by $\mathscr{C}$ the set of coidempotent subcoalgebras of D. Now we are allowed to define two maps Φ and Ψ as follows:

- Φ: Ł $\longrightarrow \mathscr{C}$; $(U, \psi) \mapsto \operatorname{Ker}\psi$, and
- $\Psi: \mathscr{C} \longrightarrow$ Ł; $A \mapsto S_AT_A(D)$, where S_AT_A is the localizing functor associated to A.

Theorem 2.5. *Let D be a coalgebra. The maps Φ and Ψ defined as above are isomorphisms and inverse to each other.*

Proof. Given a coidempotent subcoalgebra A of D, we have to show that $\Psi\Phi(A) = A$. Let S_AT_A be the localizing functor with respect to the torsion theory $\mathscr{T}_A$, and let

$\psi: D \longrightarrow U = S_A T_A(D)$ be the representing bicolinear D-map. We have to show that $\mathrm{Ker}\,\psi = A$. We know that $\mathrm{Ker}\,\psi$ is a subcoalgebra of D which is torsion, i.e, a right A-comodule. This implies that $\mathrm{Ker}\,\psi \subseteq A$. On the other hand, $\mathrm{Ker}\,\psi$ is the maximal torsion subcomodule of D since ψ is the adjunction map. But A is obviously a torsion subcomodule of D. It follows that $A \subseteq \mathrm{Ker}\,\psi$. Therefore, $A = \Psi\Phi(A)$.

Conversely, suppose that (U, ϕ) is a localization bicomodule of D. $A = \mathrm{Ker}\phi$ is a coidempotent subcoalgebra. Let $S_A T_A$ be the localizing functor with respect to the torsion theory $\mathscr{T}_A$. Let $\psi: D \longrightarrow S_A T_A(D)$ be the adjunction map with which $\mathrm{Coker}\,\psi$ and $\mathrm{Ker}\,\psi$ are torsion. By Lemma 2.2, $\psi \,\square_D\, U$ is an isomorphism. On the other hand, $(S_A T_A(D), \psi)$ is a localization bicomodule, and $\mathrm{Ker}\,\phi$, $\mathrm{Coker}\,\phi$ are torsion by Lemma 2.2, we have that $S_A T_A(D) \,\square_D\, \phi$ is an isomorphism by Lemma 2.2. This gives bicolinear isomorphism from U to $S_A T_A(D)$, and hence $\Phi\Psi([U]) = [U]$, where $[U]$ represents the equivalence class of U. □

A localization bicomodule (U, ψ) is called a left *perfect localization* if U_D is quasi-finite and injective as a right D-comodule.

Let (U, ψ) be a left perfect localization. Since U_D is quasi-finite and injective, by [5, 2.5] we may associate an MT-context to U_D

$$(C,\, D,\, {}_C U_D,\, {}_D Q_C,\, f,\, g)$$

such that $f: C \xrightarrow{\simeq} U \,\square_D\, Q$, where $C = e_{-D}(U_D)$. The bicomodule structure of ${}_D U_D$ induces a coalgebra map $\phi: C \longrightarrow D$. It is easy to check that $\psi \,\square_D\, C : C \longrightarrow U \square_D C$ is the following composite D-bicolinear isomorphism:

$$C \xrightarrow{f} U \,\square_D\, Q \xrightarrow{\psi \square U \square Q} U \,\square_D\, U \,\square_D\, Q \xrightarrow{U \square f^{-1}} U \,\square_D\, C.$$

We show that ϕ is a left coflat monomorphism

Lemma 2.6. *Let (U, ψ) be a left perfect localization and let $C = e_D(U), \phi: C \longrightarrow D$ the induced coalgebra map. Then $(C, D, {}_C U_D, {}_D C_C, F, G)$ is an MT-context and F is an isomorphism, where $F = \psi \,\square_D\, C$, $G = \psi$.*

Proof. It is enough to show that F, G are compatible. One may see that $U \,\square_D\, G = F \,\square_C\, U$ follows from the fact that $U \,\square_D\, \psi = \psi \,\square_D\, U$. To check that $C \,\square_C\, F = G \,\square_D\, C$, we compute that, $c \in C$,

$$\begin{aligned}(C \,\square_C\, F)(c) &= \sum c_{(1)} \,\square_C\, F(c_{(2)}) \\ &= \sum c_{(1)} \,\square_C\, \psi\phi(c_{(2)}) \,\square_D\, c_{(3)} \\ &= \sum \psi\phi(c_{(1)}) \,\square_D\, c_{(2)} \\ &= (G \,\square_D\, C)(c). \qquad \square\end{aligned}$$

Proposition 2.7. *Let $\phi : C \longrightarrow D$ be the coalgebra map as above. Then ϕ is a left coflat monomorphism.*

Proof. That ϕ is a left coflat map follows from Lemma 2.6 and [5, Theorem 2.5]. Let X be in ${}^C\mathbf{M}$, we have an isomorphism in ${}^C\mathbf{M}$

$$\psi \,\square_D X : X \longrightarrow U \,\square_D X,$$

since $(\psi \,\square_D C) \,\square_C X$ is a left C-isomorphism. If $X, Y \in {}^C\mathbf{M}$ and $h : X \longrightarrow Y$ is a D-comodule map, then it is a C-comodule map since the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\psi \square_D X} & U \square_D X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle U \square_D h} \\ Y & \xrightarrow{\psi \square_D Y} & U \square_D Y \end{array}$$

Hence the functor $(-)_\phi : {}^C\mathbf{M} \longrightarrow {}^D\mathbf{M}$ is full so that ϕ is a monomorphism by [2, Theorem 3.5]. □

3. Coflat monomorphisms

In this section, we investigate left coflat monomorphisms of coalgebras, and establish a bijective correspondence between left coflat monomorphisms and left perfect localization bicomodules. First, we have an easy observation: the functor of direct sum preserves coflat monomorphisms. That is,

Proposition 3.1. *If $F_i : C_i \longrightarrow D_i$ are left coflat monomorphisms of coalgebras, then $\bigoplus_i : \bigoplus_i C_i \longrightarrow \bigoplus_i D_i$ is a left coflat monomorphism.*

Proof. Straightforward. □

Lemma 3.2. *Let $\phi : C \longrightarrow D$ be a left coflat monomorphism of coalgebras, and ${}_A\mathcal{T}$ the corresponding torsion theory.*

(1) *Any left C-comodule X is torsion free as a left D-comodule.*
(2) *A comodule ${}_CX$ is C-injective if and only if ${}_DX$ is injective as a D-comodule.*
(3) *The torsion theory ${}_A\mathcal{T}$ is cogenerated by ${}_DC$.*

Proof. It is enough to show that ${}_DC$ as a left D-comodule is torsion free since any left C-comodule as a left D-comodule is still cogenerated by ${}_DC$. To show that ${}_DC$ is torsion free, it is sufficient to check that $\mathrm{Com}_{D-}(F, C) = 0$ for any finite dimensional object $F \in {}_A\mathcal{T}$ because ${}_A\mathcal{T}$ is hereditary and any comodule is locally finite. But we have

$$\mathrm{Com}_{D-}(F,C) \simeq F^* \square_D C = 0,$$

where F^* is a right A-comodule in $\mathscr{T}_A$.

(2) Suppose that ${}_CX$ is an injective comodule. Then there exists some set I such that ${}_CX \oplus_C Y = C^{(I)}$. Observe that the restriction functor $(-)_\phi : {}^C\mathbf{M} \longrightarrow {}^D\mathbf{M}$ is exactly the cotensor functor $C \square_C -$ which is exact. It follows that ${}_DX \oplus_D Y =_D C^{(I)}$. Now since ${}_DC^{(I)}$ is injective ${}_DX$ is injective. Conversely, if a left C-comodule X as a left D-comodule is injective, then there are some set J and D-comodule Z such that ${}_DX \oplus Z = D^{(J)}$. Now after cotensoring by C_D we obtain

$$C \square_D X \oplus C \square_D Z \simeq C^{(J)}$$

as left C-comodules. But ϕ is a monomorphism. By [2, Theorem 3.5] ${}_CX \simeq C \square_D X$ as left C-comodules. It follows that ${}_CX$ is injective.

(3) Follows from the proof of (1). □

Let $\phi : C \longrightarrow D$ be a left coflat monomorphism. Let $\mathscr{T}_A$ be the kernel of $- \square_D C$. By Theorem 2.5 we may identify A with (U, ψ), where $U = S_A T_A(D), \psi = \psi_D$. In this case, the D-bicomodule C plays the role of U in Lemma 2.2. That is, if $f : X \longrightarrow Y$ is a map in $\mathbf{M}^D$, then $\mathrm{Ker} f, \mathrm{Coker} f$ are in $\mathscr{T}_A$ iff $f \square_D C$ is an isomorphism. It follows from $C \square_D C \cong C$ that $\mathrm{Ker}\phi$ and $\mathrm{Coker}\,\phi$ are torsion relative to $\mathscr{T}_A$.

Lemma 3.3. *Let $\phi : C \longrightarrow D$ be a left coflat monomorphism.*

(1) U may be furnished into a (C,D)-bicomodule.

(2) $f_A = \psi \square_D C : C \longrightarrow U \square_D C$ is a C-bicolinear isomorphism.

Proof. By forgoing remark, $\mathrm{Ker}\phi$ and $\mathrm{Coker}\,\phi$ are torsion. It follows from Lemma 2.2 that $\phi \square_D U : C \square_D U \longrightarrow U$ is a D-bicolinear isomorphism, which gives U a left C-comodule structure. The second statement follows immediately from the fact that $\psi \square_D C$ factors as

$$\psi \square_D C : C \xrightarrow{\Delta} C \square_D C \xrightarrow{C \square_D \psi \square_D C} C \square_D U \square_D C \xrightarrow{\phi \square_D U \square_D C} U \square_D C. \qquad \square$$

Now let $f_A = \psi \square_D C$ and $g_A = \psi$. It is easy to see that $f_A \square_C U = U \square_D g_A$ follows from $\psi \square_D U = U \square_D \psi$ (because (U, ψ) is a localization). $C \square_C f_A = g \square_D C$ is trivial. Thus we have proved the following:

Theorem 3.4. *Let $\phi : C \longrightarrow D$ be a left coflat monomorphism. Then $(C, D, {}_CU_D, {}_DC_C, f_A, g_A)$ forms a Morita–Takeuchi context. Moreover, f_A is an isomorphism and $\mathrm{Ker} g_A = A$.*

Corollary 3.5. *Let $\phi : C \longrightarrow D$ be a left coflat morphism, U as above. Then*

(1) ${}_DC$ is a quasi-finite (injective) comodule.

(2) $C \cong e_{D-}(C) \cong e_{-D}(U)$ *as coalgebras.*
(3) (U, ψ) *is a left perfect localization bicomodule.*

Proof. It follows from [5, Theorem 2.5]. □

Now we are able to establish a correspondence between left perfect localizations and left coflat monomorphisms. Two coalgebra maps $u : C \longrightarrow D$ and $v : E \longrightarrow D$ are *isomorphic* if there is a coalgebra isomorphism $h : C \longrightarrow E$ such that $u = vh$.

Theorem 3.6. *Let D be a coalgebra. There is one-to-one correspondence between the set of isomorphism classes of left coflat monomorphisms $\phi : C \longrightarrow D$ and the set of equivalence classes of left perfect localizations (U, ψ) in ${}^D\mathbf{M}$.*

Proof. Let $\phi : C \longrightarrow D$ be a left coflat monomorphism. By Corollary 3.5, $(U = S_A T_A(D), \psi_D)$ is a left perfect localization bicomodule, and $C' = e_{-D}(U) \cong C$. Let $\phi' : C' \longrightarrow D$ be the induced coalgebra map by bicomodule ${}_D U_D$. It is clear that ϕ' is isomorphic to ϕ.

Conversely, let (U, ψ) be a left perfect localization. Let $\phi : C \longrightarrow D$ be the resulted left coflat monomorphism in Proposition 2.7, where $C = e_{-D}(U)$. Let $A = \mathrm{Ker}\,\psi$ and A' be the coidempotent subcoalgebra corresponding to the torsion theory $\mathrm{Ker}(- \,\square_D C)$. To show that (U, ψ) is equivalent to $(S_A T_A, \psi_D)$, it is equivalent to show $A = A'$ by Theorem 2.5. Since coidempotent subcoalgebras bijectively correspond to hereditary torsion theories in $\mathbf{M}^D$, it is enough to show $\mathscr{T}_A = \mathscr{T}_{A'}$. In fact, if $X \in \mathbf{M}^D$, $X \square_D C = 0$ implies $X \square_D U = (X \square_D C) \square_C U = 0$. Conversely, if $X \square_D U = 0$, then $X \square_D C \simeq X \square_D (U \,\square\, Q) = 0$, where $Q = h_{-D}(U, D)$. Thus we obtain that $X \square_D U = 0$ iff $X \square_D C = 0$. That is, $\mathscr{T}_A = T_{A'}$. □

Now we are able to show which coidempotent subcoalgebras correspond to left coflat monomorphisms.

Corollary 3.7. *Let D be a coalgebra. There is a one-to-one correspondence between the set of isomorphism classes of left coflat monomorphisms $\phi : C \longrightarrow D$ and the set of coidempotent subcoalgebras A such that D/A is quasi-finite as a right D-comodule and the localizing functor ${}_A S_A T$ (equivalently ${}_A S$) is exact.*

Proof. It is enough to show that a localization bicomodule (U, ψ) is perfect if and only if D/A is quasi-finite and ${}_A S_A T$ is exact for coidempotent subcoalgebra $A = \mathrm{Ker}\,\psi$. By Lemma 2.2 $({}_A S_A T(D), \psi')$ is equivalent to $(\simeq S_A T_A(D), \psi)$. Since ${}_A S_A T$ is isomorphic to the cotensor functor ${}_A S_A T(D) \square_D -$, ${}_A S_A T$ is exact if and only if ${}_A S_A T(D)$ is injective as a right D-comodule. In this case $S_A T_A(D) \simeq {}_A S_A T(D)$ is a injective hull of D/A. So $S_A T_A(D)$ is quasi-finite if and only if D/A is quasi-finite. □

Remark 3.8. The categorical translation of quasi-finiteness of D/A is that the canonical functor T_A has a left adjoint functor. Indeed, T_A is isomorphic to $T_A(D) \square_D -$. So T_A

has a left adjoint functor if and only if $T_A(D)$ is quasi-finite in $\overline{\mathbf{M}^D}$ which is abelian category of finite type. For a torsion-free comodule $X \in \mathbf{M}^D$, $T_A(X)$ is quasi-finite iff X_D is quasi-finite. Since $T_A(D) \simeq T_A(D/A)$ and D/A is torsion free. It follows that T_A has a left adjoint functor iff D/A is quasi-finite as a right D-comodule.

Finally, we will see a left coflat monomorphism $\phi : C \longrightarrow D$ determines a categorical equivalence between ${}^C\mathbf{M}$ and $\overline{{}^D\mathbf{M}}$ as well as it results in an equivalence between $\mathbf{M}^C$ and $\overline{\mathbf{M}^D}$ [2]. Consider the hereditary torsion theory ${}_A\mathcal{T}$ in ${}^D\mathbf{M}$. We may form the quotient category ${}^D\mathbf{M}/{}_A\mathcal{T}$, denoted by $\overline{{}^D\mathbf{M}}$. Let $({}_AT, {}_AS)$ be the canonical adjoint pair of functors between ${}^D\mathbf{M}$ and $\overline{{}^D\mathbf{M}}$. The following proposition is the left version of the equivalence in [2, Theorem 6.1].

Proposition 3.9. *Let $\phi : C \longrightarrow D$ be a left coflat monomorphism. The following composite functors define an equivalence between ${}^C\mathbf{M}$ and $\overline{{}^D\mathbf{M}}$:*

$$ {}^C\mathbf{M} \underset{U\Box_D -}{\overset{(-)_\phi}{\rightleftarrows}} {}^D\mathbf{M} \underset{{}_AS}{\overset{{}_AT}{\rightleftarrows}} \overline{{}^D\mathbf{M}}. $$

Proof. Let $F = {}_AT \circ (-)_\phi$. It is clear that F is a left exact functor and preserves direct sums. So $F \simeq F(C) \Box_C -$. Since ${}_DC$ is a torsion free and injective object we have that ${}_DC \simeq {}_AS_AT(C)$. Let $V = {}_AT({}_DC)$. We claim that V is a cogenerator in $\overline{{}^D\mathbf{M}}$. In fact, $\forall X \in \overline{{}^D\mathbf{M}}$, there is a non-torsion C-comodule X' such that ${}_AT(X') = X$. Since ${}_DC$ cogenerated the torsion theory ${}_AT$, we have

$$ \mathrm{Hom}(X, V) = \mathrm{Hom}({}_AT(X'), V) \simeq \mathrm{Com}_{D-}(X', \ {}_AS(V)) \simeq \mathrm{Com}_{D-}(X', C) \neq 0, $$

where Hom are taken in $\overline{{}^D\mathbf{M}}$. Since ${}_DC$ is torsion free and injective, V is an injective object in $\overline{{}^D\mathbf{M}}$. A similar argument shows that V is a quasi-finite object in $\overline{{}^D\mathbf{M}}$. Now we may copy the argument of [5, 5.1–5.11] to get an equivalence between $\overline{{}^D\mathbf{M}}$ and the comodule category ${}^E\mathbf{M}$, where $E = h_{\overline{{}^D\mathbf{M}}}(V, V)$ is a coalgebra and the cohom functor $h_{\overline{{}^D\mathbf{M}}}(V, -)$ induces the equivalence. By the adjoint isomorphism and Corollary 3.5 we have

$$ h_{\overline{{}^D\mathbf{M}}}(V, V) \cong h_{D-}(C, C) \cong C. $$

It is clear that the inverse functor of the cohom functor is exactly the cotensor functor $V \Box_C -$. But $V = F(C)$ and hence $F = V \Box_C -$. It remains to show that the cohom functor $h_{\overline{{}^D\mathbf{M}}}(V, -)$ is isomorphic to the composite functor $G = (U \Box_D -) \circ {}_AS$. It is enough to show that G is a left inverse of functor F. Indeed, for any $M \in {}^C\mathbf{M}$,

$$ {}_CM \simeq C \Box_C M \simeq U \Box_D C \Box_C M \simeq U \Box_D M. $$

View M as a left D-comodule. We have an exact sequence

$$ 0 \longrightarrow \mathrm{Ker}\psi_M \longrightarrow M \xrightarrow{\psi_M} {}_AS_AT(M) \longrightarrow \mathrm{Coker}\,\psi_M \longrightarrow 0. $$

Note that a left D-comodule X is torsion if and only if $U \square_D X = 0$ Applying the exact functor $U \square_D -$ to the above exact sequence, we arrive at the isomorphisms:

$$_CM \simeq U \square_D M \simeq U \square_D, \qquad {}_AS_AT(_DM) = GF(M),$$

and the proof is complete. □

Acknowledgements

We thank the referee for his valuable suggestions. His comments have led to the localizations included here. The third author would like to thank the Dept. of Algebra and Analysis for the warm hospitality when he visited the University of Almeria where this research was done.

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